Damage spreading and Lyapunov exponents in cellular automata*

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Abstract

Using the concept of the Boolean derivative we study damage spreading for one dimensional elementary cellular automata and define their maximal Lyapunov exponent. A random matrix approximation describes quite well the behavior of "chaotic" rules and predicts a directed percolation-type phase transition. After the introduction of a small noise elementary cellular automata reveal the same type of transition.

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1 Introduction

The behavior of the distance between two configurations submitted to the same dynamics (damage spreading) is considered to be a good tool to investigate the ergodic properties of the dynamics of discrete statistical models [1]. Although the relation between these properties and "chaotic" behavior is still unclear, there is an intuitive connection between "chaos" and damage spreading on one side, and between a periodic attractor and damage collapsing on the other. For continuous dynamical systems a positive maximal Lyapunov exponent (MLE) implies chaotic motion. The MLE is roughly defined as the rate of the exponential divergence of the distance between two initially close trajectories, in the limit of long times and vanishing initial distances. In what follows we show how Boolean derivatives may be used to define the MLE of a cellular automaton.

A Boolean one-dimensional cellular automaton (CA) is a discrete dynamical system defined on a lattice. The state of the system is represented by a configuration $\mathbf{x} = (x_1, \dots, x_i, \dots, x_L)$ of Boolean variables, where L is the size of the lattice. We always use periodic boundary conditions ($x_{i+L} = x_i$). The time evolution of the system is given by a deterministic Boolean function $\mathbf{F}$

$$\mathbf{x}(t+1) = \mathbf{F}\left(\mathbf{x}(t)\right), \tag{1}$$

which is in turn defined by a local, uniform rule f

$$x_i(t+1) = f(x_{i-r}, \dots, x_i, \dots, x_{i+r}); \tag{2}$$

where r is the range of the function f. There are 2^{2r+1} different CA of range r. In what follows we restrict our study to elementary CA for which $r = 1$, using Wolfram's labeling convention [2].

In the context of CA where time, space and dynamical variables are discrete, we cannot extend directly the definition of Lyapunov exponents [3, 4, 5, 6, 7, 8]. Due to the finite interaction range r and to the finite number of states of the variables $x - i$, the distance between two initially close configurations can increase at most linearly for long times.

Instead of looking at the long time behavior of the distance between two configurations, we can use some hints from the theory of continuous dynamical systems and study the local stability of a single trajectory with respect to a small perturbation (a damage or defect in the configuration). This defect can be readily recovered, or it can freeze being replicated without change, or finally it can propagate increasing the distance between the configurations.

In section 2, we show that the distance between two configurations after the introduction of a defect is given by the Boolean Jacobian matrix of the evolution function $\mathbf{F}$. While in the actual evolution of the automaton the defects can interact and annihilate themselves, we are interested in the stability of a single trajectory, and we restrict to the case of non-interacting defects which is equivalent to consider a product of Boolean Jacobian matrices on a trajectory. For the elementary CA it is a Jacobi matrix with elements equal to zero or one on the three main diagonals. This in turn suggests a relation with the product of random matrices of the same type. The MLE of the product of these random matrices shows a transition related to that of directed percolation [11, 12].

As reported in section 3, the results of simulations of "chaotic" CA (whose space-time patterns, starting from a random configuration are disordered and aperiodic) agree quite well with the predictions of the random matrix approximation. Adding a noise on the evolution of the automaton, our approach reveals the existence of a transition in the space of CA rules from a "frozen phase" where damage does not spread, to a phase where damage spreads locally with a positive MLE close to the one given by the product of random matrices.

2 Boolean derivatives, defects and random matrices

We are interested in the local stability with respect to a small perturbation, of the time evolution (1) of the configuration $\mathbf{x}$. Let us denote a defect at site i the configuration $\mathbf{z}^{(i)}$ with elements $z_j = \delta_{i,j}$, $j = 1, \ldots, L$ and $\delta_{i,j}$ the usual Kronecker symbol. The configuration $\mathbf{y}(t) = \mathbf{x}(t) \oplus \mathbf{z}^{(i)}$ differs from $\mathbf{x}(t)$ only at site i, where the XOR operation ($\oplus$) is performed site by site. Depending on $\mathbf{F}$ and on the configuration $\mathbf{x}$, the defect $\mathbf{z}^{(i)}$ can originate in one time step up to three defects in sites $i-1$, i and $i+1$. We write

$$\mathbf{x}(t+1) \oplus \mathbf{y}(t+1) = F'_{i-1,i} \wedge \mathbf{z}^{(i-1)} \oplus F'_{i,i} \wedge \mathbf{z}^{(i)} \oplus F'_{i+1,i} \wedge \mathbf{z}^{(i+1)};$$

where $F'_{i,j} = 0, 1$ and the symbol $\wedge$ represents the AND operation, to be performed between the number $F'_{i,j}$ and each element of the configuration $\mathbf{z}$. The quantities

$$F'_{i,j} = \frac{\partial x_i(t+1)}{\partial x_j(t)}.$$

are the elements of the Boolean Jacobian matrix $\mathbf{F}'$ of $\mathbf{F}$; they are expressed by the Boolean derivatives of the local evolution rule f of eq. (2) [9, 10]. For instance

$$\frac{\partial x_i(t+1)}{\partial x_{i+1}} = f(x_{i-1}, x_i, x_{i+1}) \oplus f(x_{i-1}, x_i, , x_{i+1} \oplus 1).$$

As f has range 1, $\partial x_i(t+1)/\partial x_j$ vanishes if $|i-j| > 1$, and $\mathbf{F}'$ is a Jacobi matrix whose elements are zero or one. If the local evolution rule is expressed in terms of AND and XOR operations (ring sum expansion), the Boolean derivatives extract the linear part of f.

We are interested in the limit of a small initial perturbation to a given trajectory. This limit corresponds in discrete dynamics to the presence of only one point defect. If, during the evolution, m defects appear, we consider m of replicas of the system and assign one of the defects to each one. We indicate with $N_i(t)$ the number of replicas carrying the defect $\mathbf{z}^{(i)}$ at time t. If for instance we start at time zero with only one defect at some site i ($N_i(0) = 1$), and the rule allows the spreading of the defects to the sites of the neighborhood at each time step, at $t = 1$ $N_{i-1} = N_i = N_{i+1} = 1$, at $t = 2$ $N_{i-2} = N_{i+2} = 1$, $N_{i-1} = N_{i+1} = 2$, $N_i = 3$, etc.

The time evolution of the number of defects at site i is given by

$$N_i(t+1) = \sum_j F'_{i,j}(t) N_j(t),$$

or, in matrix form,

$$\mathbf{N}(t+1) = \mathbf{F}'\mathbf{N}(t), \tag{3}$$

where the elements of $\mathbf{F}'$ are now interpreted as integer numbers. It is worth noting that $N_i(t)$ is also the number of paths in defect space that reach the site i at time t starting from any defect at time $t = 0$.

We define the finite-time MLE $\lambda(T)$ of the mapping (3) as

$$\lambda(T) = \frac{1}{T}\sum_t \log \eta(t). \tag{4}$$

where the local expansion rate of defects η is defined as

$$\eta(t) = |\mathbf{N}(t+1)|/|\mathbf{N}(t)|, \tag{5}$$

and $|\mathbf{N}| = \sum_i N_i$. In the following $\lambda(\infty)$ will be denoted simply by λ. This definition is meaningful because the number $|\mathbf{N}(t)|$ can diverge exponentially.

If $\lambda < 0$ the number of defects (the damage) decreases exponentially to zero, while if $\lambda > 0$ the damage spreads. Let us give some simple examples. Rule 0 that maps all the configurations to the configuration $\{0\}^L$, has $\lambda = -\infty$ because the Jacobian is zero. The "chaotic" rule 150 has $\lambda = \log 3$, because all its Boolean derivatives are equal to one. A marginal case is rule 204, for which $\mathbf{F}'$ is the identity matrix and $\lambda = 0$. The derivatives of the 88 "minimal" elementary CA may be found in ref. [9].

In the spreading case, a reasonable approximation to the dynamics of defects (3) consists in substituting the deterministic matrix $\mathbf{F}'$ with a random matrix of the same form. We therefore consider the product of random tridiagonal matrices $\mathbf{M}(p)$ having a fraction p of elements on the three principal diagonals equal to one. The quantity p is interpreted as the geometric mean μ of the derivative on the CA configuration for large T, i.e.,

$$\mu(T) = \left(\prod_{t=1}^{T} \mu(t)\right)^{1/T}$$

and

$$\mu(t) = \frac{1}{3L}\sum_{i=1}^{L}\sum_{k=-1}^{1} F'_{i,i+k}.$$

The evolution of the number of defects in the random matrix approximation defines a directed bond percolation problem with control parameter p, assuming that a site at location i at time t is "wet" if $N_i(t) > 0$. Observe that $N_i(t)$ gives the number of directed paths that reach site i at time t inside the percolating cluster. Therefore we expect a second–order phase transition at $p = p_c$ with order parameter the density of wet sites $\rho(t)$.

We have first localized the percolation threshold at $p_c = 0.441(1)$ (where the number in parenthesis is the error on the last significant digit) by looking at the asymptotic behavior of the order parameter. Then, starting with an initial condition where all the sites are wet, we have verified that $\rho(t) \sim t^{-\beta/\nu_\parallel}$ at p_c with $\beta/\nu_\parallel = 0.155(3)$ the usual exponents of directed percolation.

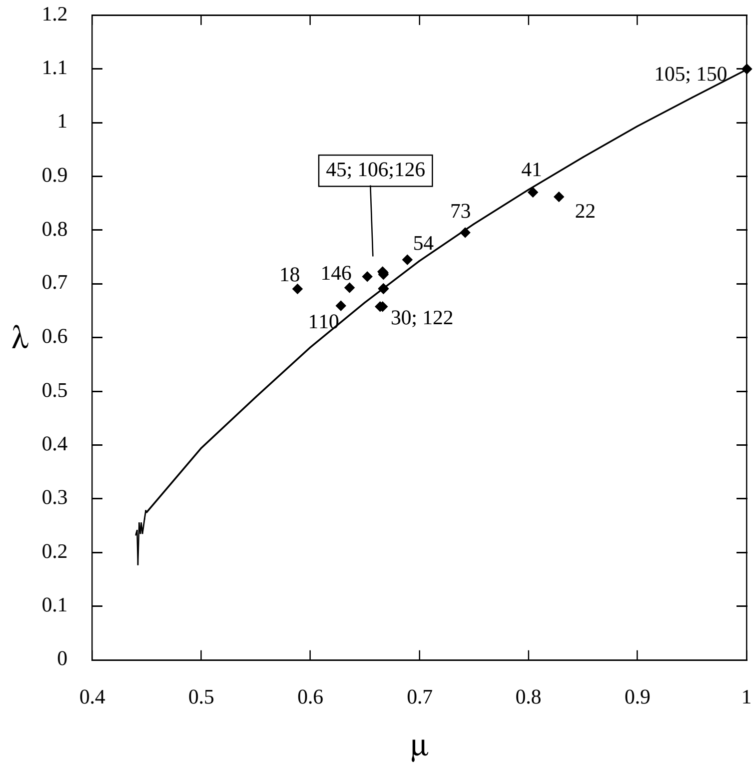

Figure 1: The curve shows the MLE λ of a random tridiagonal matrix as a function of p. The diamonds show the asymptotic value of λ for "chaotic" CA. Results for CA were obtained by letting each automaton evolve during 5000 time steps on a 512 sites lattice and $\alpha_0 = 0.5$.

The results of the random matrix approximation are reported in Fig. 1, where the curve shows the dependence of the MLE λ for the product of random matrices $\mathbf{M}(p)$ as a function of p. For $p < p_c$ $\lambda = -\infty$. At p_c, for sufficiently large T,

$$\lambda(T) = \lambda + aT^{-\chi}$$

with $\lambda = 0.237(2)$, $\chi = 0.68(4)$ and $a > 0$. This shows that at the critical point the number of walks on the percolation cluster grows exponentially with time, with an effective coordination $\exp(\lambda)$. The exponent χ, which is not usually defined in percolation, might be related to the critical exponents for directed walks [13]. The data at the percolation threshold were obtained by letting a $10^4 \times 10^4$ random tridiagonal matrix evolve during 4000 time steps for 30 realizations.

We obtain a mean field approximation replacing $\mathbf{M}$ with a constant tridiagonal matrix with elements equal to p. The corresponding MLE is $\lambda = \log 3p$ which is positive for $p \geq 1/3$. This approximation agrees well with numerical simulations for $p \geq p_c$, with a maximum deviation of 18% at $p = p_c$.

In the numerical calculation of the Lyapunov exponent λ one needs to renormalize $\mathbf{N}(t)$ [14]. This is impossible if $\mathbf{N}$ is defined over integers. However, since the Lyapunov exponents are independent of the choice of the norm [15], we let $\mathbf{F}'$ (or $\mathbf{M}$) act on some abstract "tangent" space in R^L, using the usual Euclidean norm. Applying standard methods one obtains the Lyapunov exponents related to the exponential divergence of the norm of the product of $\sqrt{\mathbf{F}'^{\dagger}\mathbf{F}}$ where $\mathbf{F}'^{\dagger}$ is the

Hermitian conjugate of $\mathbf{F}'$. The standard definition of the MLE coincides with the one of eq.(4).

3 Elementary cellular automata

We computed the mean number of ones $\mu(T)$ in the Jacobian matrix and the finite-time MLE $\lambda(T)$ for all the 88 "minimal" elementary CA for $L = 256$ and $L = 512$ and $5000 \leq T \leq 15000$ starting from random initial configurations with a fixed fraction α_0 of live sites, $\alpha_0 = L^{-1} \sum x_i(0)$. The quantities $\mu(T)$ and $\lambda(T)$ are generally already asymptotic for $T \sim 5000$; moreover they show a very weak dependence on α_0 for $0.2 \leq \alpha_0 \leq 0.8$ (only rules 6, 25, 38, 73, 134 and 154 vary more than 10% but less than 20%).

We note that

- i) CA with constant $\mathbf{F}'$ independent of the configuration (e.g. rules 0, 15, 60, 90, 150, 204) have $\lambda = \log 3\mu$ with $\mu = 0$, 1/3, 2/3 or 1.
- ii) CA for which all configurations are mapped to a homogeneous state (e.g. rules 8, 32, 40, 136) have $\lambda = -\infty$. The control parameter μ is zero. These are *class 1* CA in Wolfram's classification [3].
- iii) "Chaotic" *class 3* CA with nonconstant $\mathbf{F}'$ (e.g. rules 18, 22, 30, 41, 45, 106, 110, 122, 126, 146) have $\mu > p_c$, $\lambda > 0$ and the damage spreads.

The values of MLE for the "chaotic" CA of cases *i)* and *iii)* the value of λ agrees well with the random matrix approximation, as shown in Fig. 1. This is also trivially true for rules of case *ii*.

For the automata whose evolution leads to a nonhomogeneous periodic space pattern (e.g. *class 2* CA), λ is the logarithm of the largest eigenvalue of the product of the Jacobian matrices over the periodic state. The measured value of λ is always non negative. This suggests that the asymptotic state is unstable ($\lambda > 0$) or marginally stable ($\lambda = 0$). One can think that the "freezing" of the evolution occurs because there are no "close" configurations which can be used as an intermediate state towards a more stable state. Therefore, we "heated" the evolution by exchanging the states of a small number s of pairs of randomly chosen sites at each time step. We observe that all the rules which had $\mu < 1/3$ (the critical point in the mean field approximation) relax to $\mu = 0$ and $\lambda = -\infty$, those with $1/3 < \mu < p_c$ go to $\lambda = 0$ and the rules with $\mu > p_c$ approach the random matrix result. In Fig. 2, we show the location of all the 88 minimal elementary rules with noise in the (μ, λ) diagram, starting with $\alpha_0 = 0.5$.

After the introduction of noise, the CA rules can be divided roughly in three groups, according to the value of their MLE. In the first group, with $\lambda = -\infty$ and $\mu = 0$ we find all class 1 CA (rules 0, 8, 32, 40, 128, 136, 160 and 168) and some class 2 CA (rules 1, 3, 5, 7, 11, 13, 14, 19, 23, 43, 50, 72, 77, 104, 142, 178, 200 and 232). Rules 50, 77 and 178 show very long transients of the order of $15,000$ time steps. Rule 232, a majority rule, illustrates well a typical behavior. Configurations $\{0\}^L$ and $\{1\}^L$ are fixed points for this rule. A single defect in these configurations is

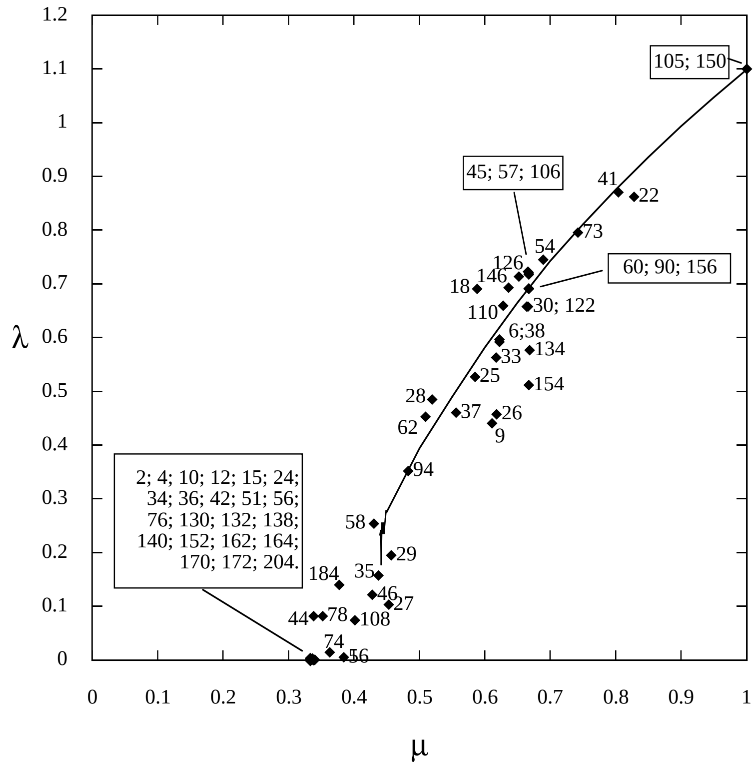

Figure 2: The curve is the same as the one shown in Fig. 1. The diamonds show the values of μ and λ for all the minimal CA with $\lambda \geq 0$.

recovered in one time step. On the other hand, an arbitraty initial configuration will give in a few time steps a pattern of strips. By adding a noise as described above, the borders of the strips perform a sort of random motion, thus allowing their merging. Finally, one of the two fixed points is reached, according to the initial density of the configuration.

In the second group we find the "chaotic" *class 3* rules. The values of μ and λ are slightly affected by the noise. They have $\mu > p_c$ and are close to the curve of the random matrix approximation for λ. We also find CA which are not *class 3* (rules 6, 9, 25, 26, 27, 28, 29, 33, 37, 38, 54, 57, 62, 73, 94, 134, 154, and 156) but show local damage spreading.

The third group contains exceptions to the random matrix approximation. This occurs for rules with a value of μ without noise close to 1/3 (CA 15, 51, ...204) or p_c (CA 1,3,5,11,14,43,142,24,44,46,56,74,108 and 152) Contrary to the prediction of the random matrix approximation $\mathbf{N}$ does not vanish. Moreover, it should be remarked that CA with a large number of conserved additive quantities (rules 3, 4, 5, 10, 12, 15, 34, 42, 51, 76, 138, 140, 170, 200 and 204) [16] have indeed $\mu = 1/3$ and $\lambda = 0$.

In this letter we have shown how the maxumum Lyapunov exponent can be defined for CA using the Boolean derivative. A positive Lyapunov exponent is associated to local damage spreading and on the other hand reflects the exponential growth of paths on the percolation clusters. For CA with $0 < \mu < p_c$ and a positive Lyapunov exponent the introduction of a small noise produces the collapse to $\lambda = 0$ or $\lambda = -\infty$. A random matrix model is directly suggested by the CA dynamics and displays a directed percolation phase transition. A somewhat similar phase

transition is observed in CA rule space in the presence of a small amount of noise. The extension of our definition of Lyapunov exponent to other discrete systems, and possibly to probabilistic dynamics will be the subject of future investigations.

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